Solvay Brussels School Economics & Management

ULB

Financial Risk Management and Governance

The Var Measure (concept vs. implementation)

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A prior

- 50s: Markowitz
  - Identification of the risk-return relationship
  - Matching with the mean-variance criterion
    - ✓ Expectation ⇢ mean(historical returns)
    - ✓ Risk ⇢ degree of dispersion ⇢ f(average spreads around average)
- Moments of a distribution and their estimator
  - Mean $$\mu = \mathrm{E}[X] \leftarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} Obs_i$$
  - Variance $$\sigma^2 = \mathrm{E}\left[(X-\mu)^2\right] \leftarrow \hat{\sigma} = \frac{1}{n-1}\sum_{i=1}^{n}(Obs_i - \mu)^2$$
  - Skewness $$s = \frac{\mathrm{E}\left[(X-\mu)^3\right]}{\sigma^3}$$
  - Kurtosis (excess) $$k = \frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4} - 3$$

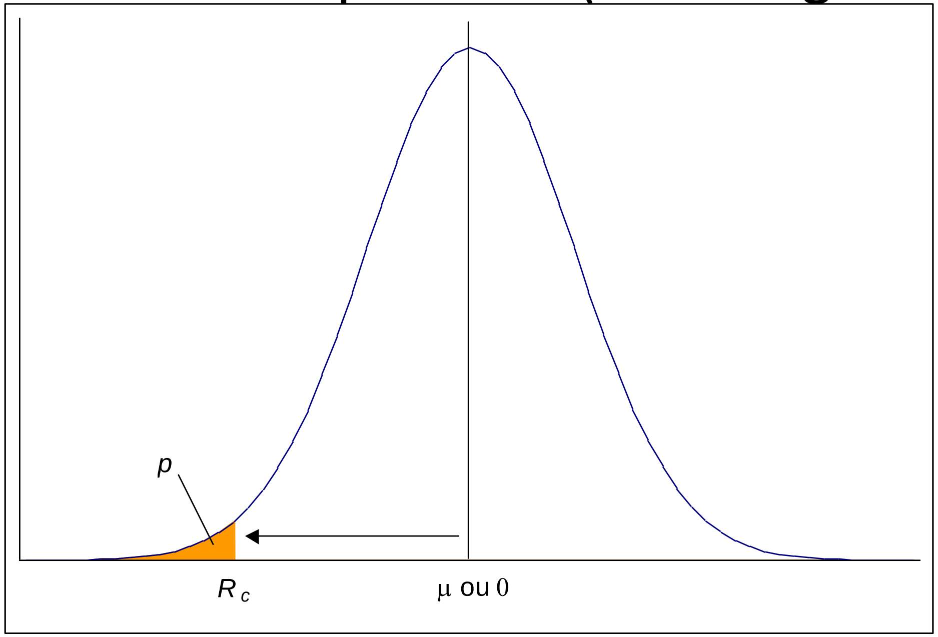
In trading activities

- We have seen many sensitivities being used and/or “greeks”

- What are the limits of their application?
  - We are not necessarily looking at one position at a time
  - We are not necessarily looking at day-trading.

The VaR concept

- Introduced in the 90s
- “Downside” risk view in currency terms
- Maximum expected loss on a given time-horizon so that the probability of higher losses is lower than a pre-specified level
- Applicable to an entire portfolio (including zero-valued positions at t=0)



The normal distribution of...returns

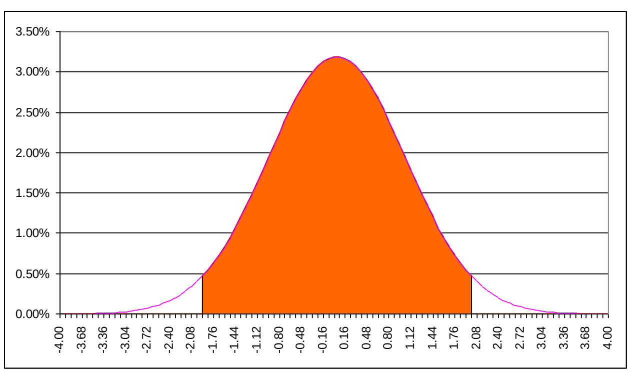
- Estimators depend on time intervals Δt
- Continuous returns $$R_t = \ln\left(P_t / P_{t-1}\right)$$
- Time « compounding » means (for continuous returns), for *T* periods of length Δt $$\mu_T = \mu T \text{ and } \sigma_T = \sigma\sqrt{T}$$
- Any historical time series can be analyzed in terms of its distribuional moments
- Assumption: Laplace normal trend theorem holds (« central limit theorem »)
  - ➔ asset returns, interest rates & forex rates follow a normal distribution
  - ➔ the space of potential realizations is continuous and follows a symmetric distribution which normal flatness guarantees a rare occurence of extreme events

The normal distribution...uses

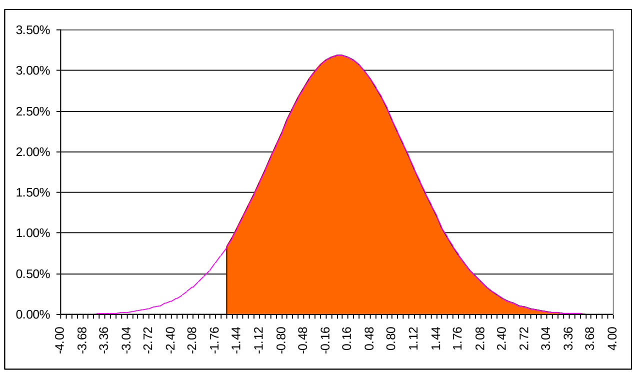
- Known theoretical distributions allow quick estimations of confidence intervals
  - The normal distributionis is only characterized by μ_T and σ_T.
  - For a scaled and centered normal variable Z (zero mean, unit variance), limit values z_c are given by

$$\Pr\left[-z_c \leq Z \leq z_c\right] = c \text{ if } Z \sim N(0,1)$$

leaving the same probability surface on each side of the pdf for a given confidence degree c.



z_c	c	p
1.645	90%	10%
1.960	95%	5%
2.241	98%	3%
2.576	99%	1%
3.291	99.9%	0.1%



z_c	c	p
1.282	90%	10%
1.645	95%	5%
1.960	98%	3%
2.326	99%	1%
3.090	99.9%	0.1%

A setting...

- A random normal can be defined as a function of *Z*

$$X = \mu_T + Z\sigma_T \qquad X \sim N(\mu_T, \sigma_T)$$

- And its confidence interval is

$$\Pr\left[\mu_T - z_c\sigma_T \le X \le \mu_T + z_c\sigma_T\right] = c$$

$$\Rightarrow \Pr\left[-z_c \le \frac{X - \mu_T}{\sigma_T} \le z_c\right] = c,$$

- Shortcomings/refinements
  - » Evidence: existence of jumps/discreteness problems
  - » Statistically: existence of leptokurtic empirical distributions

Concept (cont'd)

- A portfolio
  - Starting value: W_0
  - Expected value at $t = T$ is $W_T = W_0(1+R_T)$.
  - W_c = lowest value with a confidence degree c
  - Therefore

$$\Pr[W_T > W_c] = c \qquad \Pr[W_T \leq W_c] = p = 1 - c$$

  - Relative VaR: loss respective to W_T

$$VaR_{rel} = \mathrm{E}[W_T] - W_c = W_0(1+\mu_T) - W_0(1+R_c) = -W_0(R_c - \mu_T)$$

  - Absolute VaR: gross loss respective to W_0

$$VaR_{abs} = W_0 - W_c = W_0 - W_0(1+R_c) = -W_0 R_c$$

- Remember: assumption of normality is on « returns »

Advantages (à priori)

- Simple
- Better than…
  - » Just the exposure: VaR can be > or < than the exposure
  - » Duration, beta, option delta: VaR is sensitive to the event probability on the underlying variable
- Solution for some derivatives (for forwards and swaps….)
- Total portfolio risk
- Could be easily completed by sensitivity analysis

Examples

- Bond pricing
  - Almost sure not to loose all the value in one-week time.
  - Worst expected increase of 4y interest rate in 6 months from now, with a 95% confidence degree: 2.5%
  - 6-month VaR of a 2000€ investment in a 4y 0-coupon is

$$\begin{aligned} VaR &= \text{Amount} \times \text{Duration} \times \Delta r_{95\%} \\ &= 2000€ \times 4 \times 2.5\% \\ &= 200€ \end{aligned}$$

- Sale of options
  - We get a premium
  - Exposed to $Max(S_T - K, 0)$
  - Maximum loss can be substantially higher than its premium
  - ➔ Financial risk $\neq$ accounting of cash-ins & cash-outs
    - Ex: Baring's case & Nick Leeson

Methodologies

- Steps in examining risks
  1. Determine market value of selected position
  2. Measure sensitivity to risk sources and correlations between them
  3. Identify the time-horizon of the investment
  4. Define the confidence degree
  5. Compute the maximum expected loss
- Methods
  - var-covar
  - Historical simulations
  - MonteCarlo simulations

Comparison of models

	Delta-Normal (or var-covar)	Historical Simulation	MonteCarlo Simulation
Valuation	Linear (Local)	Full	Full
Distribution			
▪ Shape	→ Normal	→ Actual	→ General
▪ Extreme events	→ Low probability	→ In recent data	→ Possible
Implementation			
▪ Ease of computation	→ Yes	→ Intermediate	→ No
▪ Communicability	→ Easy	→ Easy	→ Difficult
▪ VaR precision	→ Excellent	→ Poor with short window	→ Good with many iterations
▪ Major pitfalls	→ Non-linearities, fat tails	→ Time variation in risk, unusual events	→ Model risk

Inspired from Jorion, *Financial Risk Manager Handbook*

Var-covar

- Making some replacements () and conscious that

$$1-c=p=\Pr\left[W_T \le W_c\right]=\Pr\left[R_T \le R_c\right]$$

- If *R* is normally distributed, then

$$\Pr\left[R_T \le R_c\right]=\Pr\left[R_T \le \mu_T - z_c\sigma_T\right]$$

- We therefore get

$$VaR_{rel}=-W_0(R_c-\mu_T)=W_0 z_c\sigma\sqrt{T}$$

$$VaR_{abs}=W_0(z_c\sigma\sqrt{T}-\mu T)$$

- The generalisation to *n* assets i and *m* risk sources j
  - » Assume: we can compute the exposure of any asset to any source of risk
  - » Exposure matrix $n \times m$

$$W_{n\times m}=\begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,m} \\ w_{2,1} & w_{2,2} & & \\ \vdots & & \ddots & \\ w_{n,1} & & & w_{n,m} \end{bmatrix}$$

Var-covar (2)

- Each column total gives a vector $W_{1\times m}^{Tot}$ of size *m*
- Compute
  - variances of each risk source
  - + covariances of all pairs, using historical data
- Expected value of portfolio

$$\mathrm{E}\left[R_p\right] = W_{1\times m}^{Tot} \mu_{m\times 1} = \begin{bmatrix} w_1^{Tot} & w_2^{Tot} & \cdots & w_m^{Tot} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix}$$

- Variance of portfolio

$$Var\left[R_p\right] = W_{1\times m}^{Tot} \Sigma_{m\times m} W_{m\times 1}^{Tot} = \begin{bmatrix} w_1^{Tot} & w_2^{Tot} & \cdots & w_m^{Tot} \end{bmatrix} \begin{bmatrix} \sigma^2{}_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,m} \\ \sigma_{2,1} & \sigma^2{}_{2,2} & & \vdots \\ \vdots & & \ddots & \sigma_{j,m} \\ \sigma_{m,1} & \cdots & \sigma_{m,j} & \sigma^2{}_{m,m} \end{bmatrix} \begin{bmatrix} w_1^{Tot} \\ w_2^{Tot} \\ \vdots \\ w_m^{Tot} \end{bmatrix}$$

- VaR

$$VaR_{rel}^{pf} = z_c \sqrt{T} \sqrt{W_{1\times m}^{Tot} \Sigma_{m\times m} W_{m\times 1}^{Tot}}$$

Var-covar (cont'd) - sensitivities

- Not all assets i present a 1 to 1 sensitivity to the risk source j
- Weights are then « scaled »
- Examples (cases):
  - » Shares considered in terms of their systematic risk to an index (and not individually) $w_{i,j} = \text{Amount} \times \beta_{\text{i,j}}$
  - » Options on an underlying present as a risk source
    $$w_{i,j} = \text{Montant} \times Delta_{\text{i,j}}$$
  - » Options (position i1) on an underlying (position i2) that is related to a risk source j. $w_{i_1,j} = \text{Montant} \times Delta_{\text{i}_1,\text{i}_2} \times \beta_{i_2,j}$
  - » Bonds
    - ✓ Duration
    - ✓ Decomposition into 0-coupon components
    - ✓ RiskMetrics™ approach → variance conservation

Var-covar (cont'd) - Indextron

- A first exercise – Data :
  - You have the following portfolio of assets (we won't explain here how you did get there)

Asset	Description	Market unit price	Currency
1	7 shares of a fund indexed on the EuroStoxx50	1500.00	EUR
2	2 shares of a fund indexed on the Dow Jones	10000.00	USD
3	10 10-years US Treasury 0-coupon bonds (face value: 1000.00 USD)	650.00	USD

  - The exchange rate USD/EUR (dollars per euro) is currently trading at 1.25
  - Your reference currency is the Euro

Var-covar (cont'd) - Indextron

- A first exercise – Q&As :
  - » Risk sources?
    - ✓ an exposure to the EuroStoxx50.
    - ✓ an exposure to the Dow Jones.
    - ✓ a currency risk exposure to USD/EUR.
    - ✓ a risk exposure to interest-rate fluctuations → Δ bond prices.
  - » Exposures?
    - ✓ Split positions into exposures
    - ✓ Allocate them to the 4 risk sources

      = « mapping »
    - ✓ Values (in €):

Pos	Description	Computation	Value in Euros
1	Shares of EuroStoxx 50 :	7 × 1500 =	10500 (33,12%)
2	Shares of Dow Jones :	(2 × 10000) / 1,25 =	16000 (50,47%)
3	US Bonds:	(10 × 650) / 1,25 =	5200 (16,40%)

Var-covar (cont'd) - Indextron

» Exposures? (cont'd)

✓ « Mappings »:

	EuroStoxx 50	Dow Jones	$/€	US 10y bonds
Shares EuroStoxx 50 :	**10500**			
Shares Dow Jones :		**16000**	**16000**	
US bonds:			**5200**	**5200**
Total	**10500**	**16000**	**21200**	**5200**

» Variances-covariances

✓ Data:

	Standard deviations	Correlations			
		EuroStoxx 50	DJ	USD	US 10 years
EuroStoxx 50	30.00%	1.00	0.49	0.64	-0.28
DJ	20.00%	0.49	1.00	0.80	-0.37
USD	10.00%	0.64	0.80	1.00	-0.43
US 10 years	9.00%	-0.28	-0.37	-0.43	1.00

Var-covar (cont'd) - Indextron

» Variances-covariances (cont'd):

✓ Knowing that $\sigma_{ij} = \sigma_i \times \sigma_j \times \rho_{i,j}$

Variances-covariances				
	EuroStoxx 50	DJ	USD	US 10y
EuroStoxx 50	0,09000	0,02940	0,01920	-0,00756
DJ	0,02940	0,04000	0,01600	-0,00666
USD	0,01920	0,01600	0,01000	-0,00387
US 10y	-0,00756	-0,00666	-0,00387	0,00810

» VaR?

✓ Multiplication of mapping-vector (with each column of) var-covar matrix (first):

EuroStoxx 50	DJ	USD	US 10y
1783,13	**1253,27**	**649,8**	**-225,86**

– Example

		EuroStoxx 50	
10500	×	0,09000	= 945,00
16000	×	0,02940	= 470,40
21200	×	0,01920	= 407,04
5200	×	-0,00756	= -39,312
		Total	1783,13

Var-covar (cont'd) - Indextron

» VaR?

✓ Second multiplication:

		EuroStoxx 50	
1783,13	×	10500	18722844,0
1253,27	×	16000	20052288,0
649,48	×	21200	13768891,2
-225,86	×	5200	-1174493,8
		Total	51369530,4

✓ Total variance is annual → weekly variance: $\sqrt{\frac{51369530{,}4}{52}} = 993{,}92$

✓ Critical value of zc for 95% confidence degree is 1,644853

✓ VaR is therefore: $993{,}92 \times 1{,}644853 = 1634{,}85$

Var-covar (cont'd) - Indextron

» Contribution to VaR?

- ✓ First derivative w.r.t. « mapping-vector » or weights (Deltas):

$$\frac{\delta VaR}{\delta W'} = z_c \sqrt{T} \frac{\Sigma W}{\sqrt{W' \Sigma W}} = z_c^2 T \frac{\Sigma W}{VaR}$$

- ✓ Interesting property:

$$VaR = W' \left(z_c \sqrt{T} \frac{\Sigma W}{\sqrt{W' \Sigma W}} \right) = \sum_{j=1}^{m} w_j \left(z_c \sqrt{T} \frac{\Sigma W}{\sqrt{W' \Sigma W}} \right)_j = \sum_{j=1}^{m} VaR_j$$

- ✓ In our case, deltas...

EuroStoxx 50	DJ	USD	US 10y
1783,13 /7167,25 *1,644853	1253,27 /7167,25 *1,644853	649,8 /7167,25 *1,644853	-225,86 /7167,25 *1,644853
0,41	0,29	0,15	-0.05

Var-covar (cont'd) - Indextron

» Contribution to VaR?

✓ In our case, component VaRs...

10500	×	0,41	= 4296,81	(36,45%)
16000	×	0,29	= 4601,91	(39,04%)
21200	×	0,15	= 3159,90	(26,80%)
5200	×	-0,05	= -269,54	(-2,29%)
		Total	11789,08	(100%)

Var-covar (cont'd) - Indextron Extensions

- How can we...
  - ...compute the amount of diversification in this portfolio?
  - ...use stocks instead of indices or index funds that are not perfectly replicating the index?

Var-covar - the cheatsheet

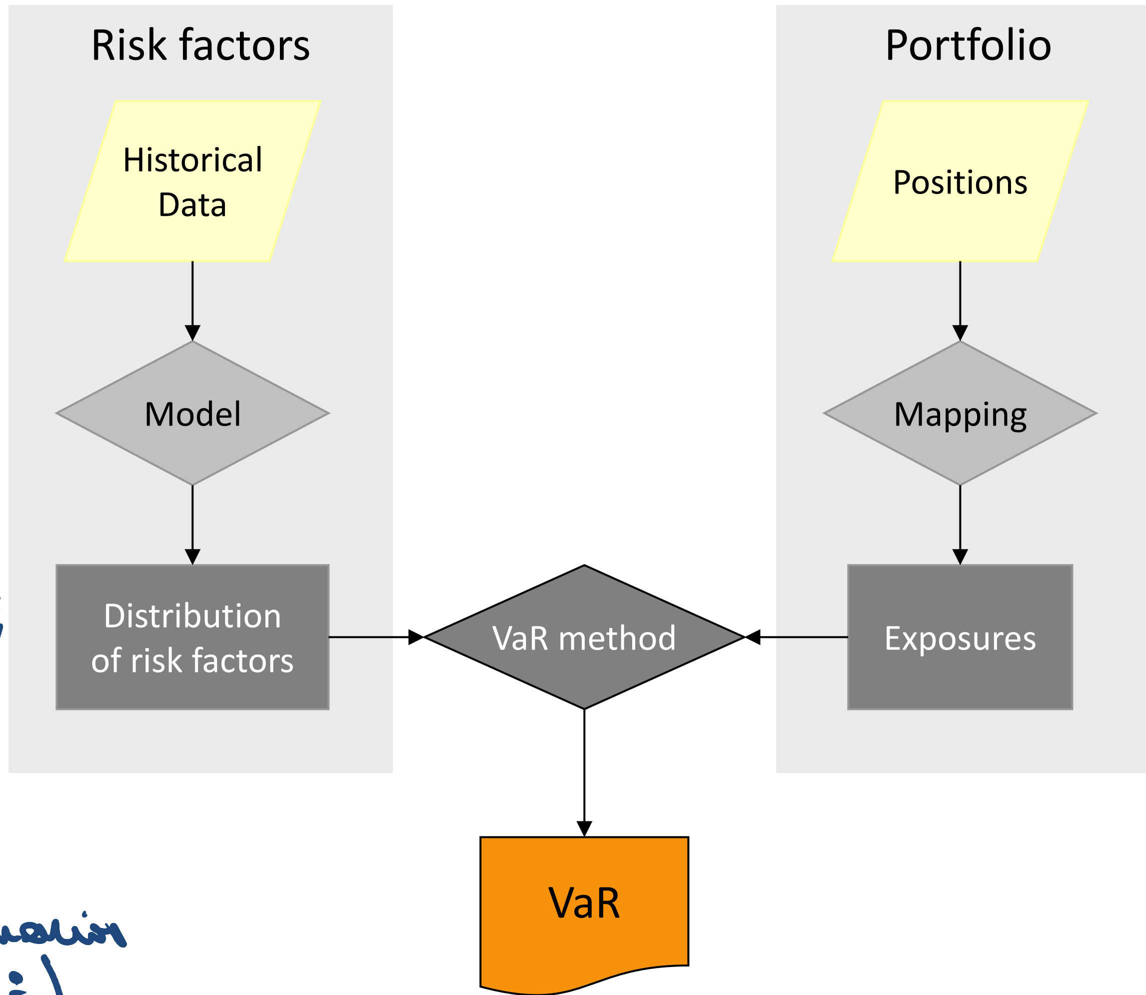
1. Compute the **current market value** of the portfolio, position by position and **identify** for each position, **the risk exposures** against your original situation and **given your reference currency**.
2. Create a «mapping» matrix: rows=positions, columns=risk exposures

$$W_{n\times m} = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,m} \\ w_{2,1} & w_{2,2} & & \\ \vdots & & \ddots & \\ w_{n,1} & & & w_{n,m} \end{bmatrix}$$

   a) For 1:1 exposures: put the amount in your reference currency
   b) For indirect exposures: pout the amount multiplied by either
      - ✓ β → stock rel. changes vs. index rel. changes
      - ✓ D → interest-rate changes vs. 0-coupon bond price rel. changes
      - ✓ Δ → option rel. changes vs. underlying rel. changes
   c) Specific case: Coupon-bearing bonds
      - ✓ Map to the closest «duration» vertex.
      - ✓ split the bond among several « risk vertices» based on PV(cash flows).
      - ✓ split the bond among several « risk vertices» based on the conservation of the total variance (RiskMetrics approach).
3. Compute statistics: volatilities and correlations
   a) Standard
   b) Or using EWMA, Arch or Garch stats
4. Compute $VaR\left[P_f, T, c\right] = z_c \sqrt{T} \sqrt{W_{1\times m}^{Tot} \Sigma_{m\times m} W_{m\times 1}^{Tot}}$
5. Nice to have:
   a) component and incremental VaRs,
   b) amount of diversification.

Summary (1)

- Ex-ante



σ, ρ for var-covar

empirical dist° (hist sim°)

σ, ρ to simulate scenarios (MC sim°)

Summary (2)

- Ex-post
  - Stress-testing
    - Scenario analysis
    - Testing models & statistical inputs
    - Developing policy responses
  - Scenarios...
    - Moving one variable at a time
      - 0-correlation
      - With correlation
    - Historical scenarios
    - Tailoring prospective scenarios
  - Goal: identify areas of potential vulnerability

Extensions

Extensions: The Quadratic Model

- Using back the idea of delta and gamma, a Taylor Series Expansion would give:
  - For the changes in a portfolio value *P*: $\Delta P = \delta\,\Delta S + \frac{1}{2}\gamma(\Delta S)^2$
  - Or $\Delta P = S\delta\,\Delta x + \frac{1}{2}S^2\gamma(\Delta x)^2$
  - Which means $\begin{cases} E(\Delta P) = 0.5\,S^2\gamma\,\sigma^2 \\ E(\Delta P^2) = S^2\delta^2\sigma^2 + 0.75\,S^4\gamma^2\,\sigma^4 \\ E(\Delta P^3) = 4.5\,S^4\delta^2\gamma\,\sigma^4 + 1.875\,S^6\gamma^3\,\sigma^6 \end{cases}$
- For *n* underlying market variables and each instrument dependent on only one of them $\Delta P = \sum_{i=1}^{n} S_i\delta_i\Delta x_i + \sum_{i=1}^{n}\frac{1}{2}S_i^2\gamma_i(\Delta x_i)^2$

  or more generally $\Delta P = \sum_{i=1}^{n} S_i\delta_i\Delta x_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}S_i\,S_j\gamma_{ij}\Delta x_i\Delta x_j$

Extension: Cornish-Fisher expansion

- Formula to approximate quantiles of a pdf based on its moments or more precisely on its "cumulants". Cumulants can be expressed in terms of its mean $\mu = E(x)$ and its central moments

$$\mu_r = E\left[(x-\mu)^r\right]$$

  - With

$$\kappa_1 = \mu$$
$$\kappa_2 = \mu_2$$
$$\kappa_3 = \mu_3$$
$$\kappa_4 = \mu_4 - 3\mu_2^2$$
$$\kappa_5 = \mu_5 - 10\mu_3\mu_2$$

  - And therefore

$$x_q \approx z_q + \frac{1}{6}\left(z_q^2 - 1\right)\kappa_3 + \frac{1}{24}\left(z_q^3 - 3z_q\right)\kappa_4 - \frac{1}{36}\left(2z_q^3 - 5z_q\right)\kappa_3^2 - \frac{1}{120}\left(z_q^4 - 6z_q^2 + 3\right)\kappa_5 - \frac{1}{24}\left(z_q^4 - 5z_q^2 + 2\right)\kappa_3\kappa_4 + \frac{1}{324}\left(12z_q^4 - 53z_q^2 + 17\right)\kappa_3^3$$

  - Applied to a normalized *x* first $x = \dfrac{x^{real} - \mu}{\sigma} \rightarrow x_q^{real} = x_q\sigma + \mu$

References

- Prof. H. Pirotte
- Some excerpts from:
  - Hull (2007), “Risk management and Financial Institutions”
  - The RiskMetrics technical document
  - Jorion (2008), “Financial Risk Manager Handbook”
- Others:
  - Jorion (2000), « Risk Management Lessons from LTCM ».